

# MAC-CPTM Situations Project

## *Situation 55: Properties of $i$ and Complex Numbers*

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### Prompt

A teacher of an algebra III course notices that her students often interpret  $i$  in expressions like  $3i$ ,  $i^2$ , and  $4i + 2$  as though it were an unknown or variable. They seem to think that because it is a letter it does not actually represent a number. The teacher wants to work with her students so that they have a feasible way of thinking about imaginary and complex numbers as being a new type of number that like other numbers have certain properties associated with them.

### Commentary

Focus 1 and 2 develop properties of  $i$  that might help to distinguish the symbol both from properties of the real numbers as well as properties of an unknown or variable. Focus 1 examines how the property of orientation is changed when  $i$  is used in multiplication as opposed to when 1 is used in multiplication. Focus 2 establishes a second salient property related to multiplication of  $i$  as opposed to multiplication by 1 that looks at how multiplication by  $i$  can be considered a rotation of 90 degrees in the counterclockwise direction. These two properties establish the two new characteristics of  $i$ . The third focus looks at how  $i$  can be considered a special case of a complex number. It suggests that the introduction of  $i$  along with the operations on the real number system is sufficient for generating the complex numbers, a number system that is sufficient for solving any quadratic equation. The fourth focus looks at how the definition of multiplication can be extended to complex numbers and connects these notions with the definition of rotation matrices and the matrix representation of complex numbers.

### Focus 1

We might think of imaginary numbers as relating to the convention of orientation. To this end, we may stipulate some conventions about orienting numbers. The first convention is that motion to the right or up from a reference value will always be considered motion in the positive direction. On the other hand, motion to the left or down from a reference value will always be considered in the negative direction. Next, when two linear dimensions produce an area we will always imagine moving in the horizontal direction first and then the vertical direction. Following this convention, area in the first quadrant will produce motion in the counterclockwise direction. We will select the counterclockwise direction to be the positive direction and the clockwise direction to be negative. Note here that we have established this orientation of the

area independently of our linear orientation. That is, we could have selected motion in the counterclockwise direction to establish a negative orientation. [See the GSP animation of unit squares in the four quadrants.](#)

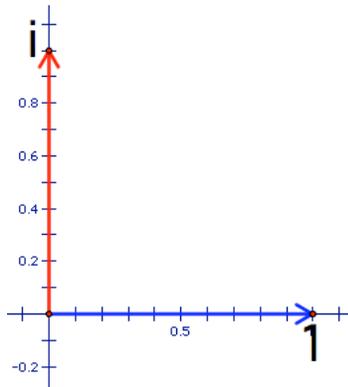
We might consider taking our two unit lengths and producing a unit area. Remember that we choose the orientation of our area independently of the orientation for the unit vectors. So this time let's consider choosing a counterclockwise direction as orienting the area negatively and a clockwise direction as orienting the area positively. We have not changed the magnitude of the area only its orientation. Therefore, the area still represents a magnitude of one but its orientation has changed. So, we have an area of  $-1$ . We now need some symbolism to represent the lengths of the sides. This symbolism cannot coincide with the symbolism we used when the orientation was positive because we want to connote that we have created something new. To do this we will call the side lengths  $i$  and we see that  $i^2 = -1$ . For our newly oriented areas, you may want to look at the GSP file again. It is worth considering what happens in each of the quadrants. For instance, in quadrant two the point moves around the area in the clockwise direction which is positive so we have that  $-i*i = 1$  (as expected). This model of an imaginary number can be used to motivate a property of imaginary numbers that differs from real numbers (and certainly distinguishes the symbol from letters like  $x$  when  $x$  stands in for a variable or unknown because prior to knowing any properties of  $x$  we need to define a domain). The property that differs from real numbers is that if we have two positively or negatively oriented numbers multiplication produces a negatively oriented number and similarly that if we have one negatively and one positively oriented number multiplication produces a positively oriented number. We can use this model to derive the common definition that  $i = \sqrt{-1}$  by taking the square root of both sides of the equation  $i^2 = -1$ .

We have now created a basic model for imaginary numbers and might want to return to situation 50 to see how imaginary solutions could be used to function in the context of factoring quadratics using an area model.

## Focus 2

Now that we have a basic model for imaginary numbers, we might want to develop a model for how to compose them additively and multiplicatively. Let's first consider multiplicative composition of  $i$  with a unit vector. We can define our multiplicative operation to be a rotation of 90 degrees in the counterclockwise direction. If this seems like a strange definition for a multiplicative operation, we can see that all other transformations of the plane are related to multiplication in various ways. For instance, dilation is a stretching by the dilation factor, translation when taken in various combinations can be seen as creating a multiplicative structure based on repeated addition, and reflection can be seen as orientation reversing or related to multiplication by  $-1$ . Further, we might make sense of this definition of multiplication by thinking of multiplication by  $-1$  as a rotation around the origin by 180 degrees in the counterclockwise direction. If we consider  $-1 = i^2$ , then it is consistent to define multiplication by  $i$  to be a rotation by half the number of degrees as multiplication by  $-1$  or  $i^2$  would produce.

In defining our multiplication in this way, we have to be clear that we are constructing a new number system because it is possible to simply consider a rotation of a unit vector as producing a unit vector that has been rotated by 90 degrees in the counterclockwise direction. However, we want to make a model for the properties of our new numbers. Therefore, we want to define this operation as being our multiplication. Note that we are making a model that is essentially an operator model. That is,  $i$  is operating on the unit vector. So the vector 1 is rotated by 90 degrees counterclockwise to create the vector  $i$  on the  $y$ -axis.



We might consider performing multiplication now on the vector  $i$ . In doing so, we get  $i^2$  which we know from focus one is the same as  $-1$ . If we perform our multiplication one more time, we have that  $i^3 = -i$  and  $i^4 = 1$  as expected. In this context, we can check that products are associative, commutative, and distributive all based on the geometric operations that are being used in particular instances. Here again, we have a new property of an imaginary number that helps us to differentiate it from symbols like  $x$  that may stand in for an unknown or variable. Certainly,  $x$  does not automatically have the property that it connotes any specific type of multiplication until we have defined a domain for the values of  $x$ .

### Focus 3

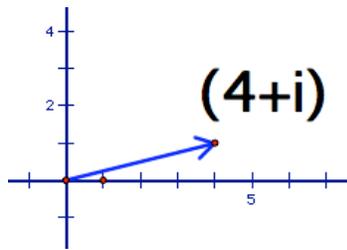
Having defined two properties that help to establish meaning for the symbol  $i$  and differentiate it from the meaning of  $x$  when  $x$  is intended to stand in for an unknown or variable, it is useful to look at the historical development of  $i$  and see that we can consider it as a special case of a complex number. Historically, mathematicians first encountered imaginary and complex numbers when factoring quadratic and cubic equations. Initially, Greeks ( $\approx 600$ -300 BCE) only recognized positive solutions to quadratic and cubic equations. However, both Hindu ( $\approx 600$  CE) and Arabian ( $\approx 1100$  CE) mathematicians introduced models for negative solutions to quadratic and cubic equations. Western mathematicians ( $\approx 1500$ ) were the first to consider complex solutions to quadratic and cubic equations. They considered these solutions at first because they came up from various algebraic solutions to such equations. Therefore, many times mathematicians think of complex numbers as being the answer to the question does every quadratic equation have a solution. The answer is yes but we need to extend the real number system. In extending the real numbers, we only need to introduce one new number,  $i$ . This new number and the operations already defined on the real numbers are sufficient for creating all

complex numbers. Because  $i$  plays this special role, the quadratic equation  $x^2 + 1 = 0$  where  $x = \sqrt{-1} = i$  is often given special status as well.

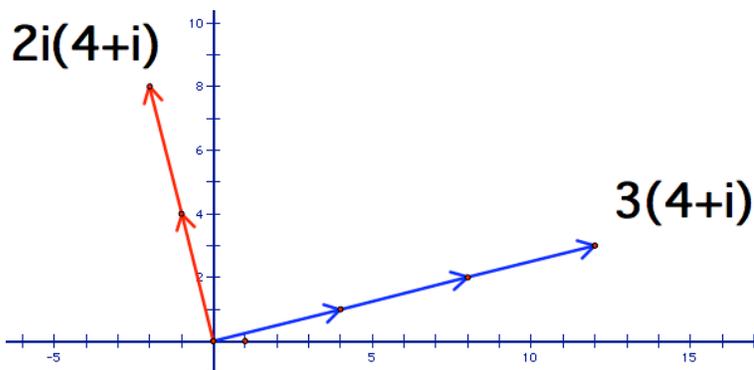
We might now consider the number  $i$  as a special case of a complex number even though it plays an essential role in generating all other complex numbers. That is,  $i$  is simply a solution to a quadratic equation that previously did not have a solution. It may be considered the same as  $3 + i$  in that this number is also the solution of a quadratic equation, namely  $x^2 + 6x + 10$ . Note that this historical development of complex and imaginary numbers was slow and fraught with controversy in part because of debates about what constituted a number. Focus 1 and 2 try to suggest that part of what constitutes a number are the properties that we might use to define or differentiate this number from other numbers or letters used in mathematics.

## Focus 4

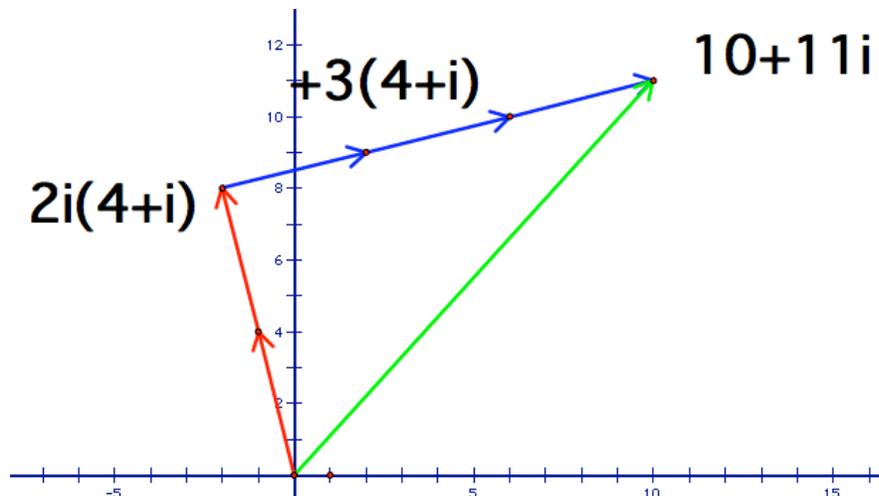
We might now want to put our notion of multiplication to the test on some complex numbers. So let's multiply  $(2i + 3)(4 + i)$ . The vector  $(4+i)$  is depicted below.



Based on our definition of multiplication when we multiply the vector  $(4 + i)$  by  $2i$  we rotate it 90 degrees counterclockwise and dilate it by 2 shown as the red vector. When we multiply the vector  $(4 + i)$  by 3 we simply dilate it by 3 shown below as the blue vector.



Finally we want to add the blue and red vector to find the new complex number. This addition is shown below as well as the resulting vector shown in green. It is possible to use geometry to derive the usual algebraic definition of multiplication of complex numbers.



Also, it is now possible to coordinate the definition of multiplication by a complex number, the matrix representation of a complex number, and the definition of a rotation matrix. A complex number is sometimes represented as a matrix in the form  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$  and a rotation matrix is defined as  $\begin{bmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{bmatrix}$  where  $\vartheta$  is the angle of rotation in the counterclockwise direction. We see

that  $bi$  would be represented by a matrix of the form  $\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  that we can interpret in terms of our rotation matrix as  $\cos \vartheta = 0$  and  $\sin \vartheta = 1$  giving that  $\vartheta = 90^\circ$  followed by a dilation of the scale factor  $b$ . This result is consistent with our earlier definition. Next, if we look at multiplication by  $a$  that  $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  giving that  $\cos \vartheta = 1$  and  $\sin \vartheta = 0$  so that  $\vartheta = 0^\circ$  or we just get a dilation by a scale factor of  $a$  again this result is consistent with our earlier definition.

## Focus 5

We might consider one further definition of complex multiplication that grows out of the notion that a complex number is a vector.